Нелинейные граничные задачи. вып.14, с.90-96, 2004.

©2004. G. Marinoschi

ON SOME PROBLEMS CONCERNING THE NONLINEAR INFILTRATION IN UNSATURATED MEDIA

The paper deals with the mathematical treatment of two specific models related to water infiltration in soils. The mathematical models consist of Richard's equation with appropriate boundary and initial conditions. The hydraulic parameters (diffusivity, hydraulic conductivity, water capacity) that represent the coefficients of this equation are nonlinear functions. Depending on the situation studied, particularities that may arise are represented by the fact that the water capacity vanishes at the saturation value, implying that the equation degenerates at the interface between unsaturated-saturated flow and diffusivity blows up at the moisture saturation value in the unsaturated model. The paper develops a theory concerning the existence of the solution of each model apart.

1. Statement of the problem.

Let Ω be an open bounded subset of $\mathbb{R}^N$ $(N = 1, 2, 3)$ with the boundary $\partial\Omega \overset{notation}{=} \Gamma$ sufficiently smooth and $(0,T)$ is a finite time interval. Let $x \in \Omega$ represent the vector $x = (x_1, x_2, x_3)$. Consider the mathematical model describing the water infiltration into an unsaturated soil

$$C(h)\frac{\partial h}{\partial t} - \nabla \cdot (K(h)\nabla h) + \frac{\partial K(h)}{\partial x_3} = f, \ (x,t) \in Q = \Omega \times (0,T) \tag{1}$$

$$h(x,0) = h_0(x), \ x \in \Omega \tag{2}$$

$$h(x,t) = g(x,t), \ (x,t) \in \Sigma = \Gamma \times (0,T). \tag{3}$$

Equation (1) is in fact Richards' equation written for the pressure head, $h(x,t)$, where $C(h)$ is the water capacity and K is the hydraulic conductivity, both depending nonlinearly on h. Given the constitutive relationship linking the volumetric water content (or moisture) θ with the pressure head, $\theta = \theta(h)$, the water capacity is derived from $C(h) = d\theta/dh$. In this study we assume that $h \to \theta(h)$ is a nonhysteretic function, continuous and monotonically increasing. The distribution of the pressures is known in $\overline{\Omega}$ at the initial time, g is known and continuous on $\Gamma \times [0,T]$ and the function f stands for some source in the domain. In the unsaturated soil, characterized by $h < 0$, the functions C and K are defined as follows: $C : [h_r, 0) \to (0, C_r]$, $K : [h_r, 0) \to [K_r, K_s)$ and they are continuous with respect to h. Here h_r is a negative number, C_r, K_r and K_s are positive numbers, all known. Moreover, C is monotonically decreasing and K is monotonically increasing on $[h_r, 0)$. In a saturated soil in which $h \geq 0$ the functions take constant values, namely, $C(h) = 0$ and $K(h) = K_s$. We assume still that

(i_K) there exists $M > 0$ such that $K'(h) \leq MC(h)$.

We extend the functions by continuity to the left of h_r, such that $0 < C(h) \leq C_r$ and $0 < K(h) \leq K_r$, for $h < h_r$, such that $\int_{-\infty}^{h_r} K(\zeta)d\zeta = +\infty$. Further we introduce the functions

$$C^*(h) = \begin{cases} \theta_r + \int_{h_r}^{h} C(\zeta)d\zeta, \ h < 0 \\ \theta_s, \ h \geq 0 \end{cases}, \ K^*(h) = \begin{cases} \int_{h_r}^{h} K(\zeta)d\zeta, \ h < 0 \\ K_s^* + K_s h, \ h \geq 0 \end{cases} \tag{4}$$

with $K_s^* = K^*(0)$, $\theta_s = C^*(0)$ and notice that $C^*(h) = \theta$. It is convenient to work in variable θ, so we introduce the inverse of C^* and define

$$h = \begin{cases} (C^*)^{-1}(\theta), \ \theta < \theta_s \\ [0, +\infty), \ \theta = \theta_s. \end{cases}, \ \beta^*(\theta) = \begin{cases} K^*((C^*)^{-1}(\theta)), \ \theta < \theta_s \\ [K_s^*, +\infty), \ \theta = \theta_s \end{cases} \tag{5}$$

and $\kappa(\theta) = K((C^*)^{-1}(\theta))$. We still denote

$$\beta(\theta) = K((C^*)^{-1}(\theta))/C((C^*)^{-1}(\theta)) \geq \rho \tag{6}$$

and we have $\beta(\theta) \geq \rho = K_r/C_r$, $\theta < \theta_s$. Also β^* satisfies

(i) $(\beta^*(\theta) - \beta^*(\overline{\theta}))(\theta - \overline{\theta}) \geq \rho(\theta - \overline{\theta})^2$, $\forall \theta, \overline{\theta} \in (-\infty, \theta_s]$

(ii) $\lim\limits_{\theta \to -\infty} \beta^*(\theta) = -\infty.$

Due to (i_K) it follows that $\theta \to \kappa(\theta)$ is Lipschitz, i.e.,

(ii_K) $\left|\kappa(\theta) - \kappa(\overline{\theta})\right| \leq M \left|\theta - \overline{\theta}\right|$, $\forall \theta, \overline{\theta} \leq \theta_s.$

Correspondingly to these definitions the mathematical model to be studied reduces to the nonlinear diffusion equation

$$\frac{\partial \theta}{\partial t} - \Delta \beta^*(\theta) + \frac{\partial \kappa(\theta)}{\partial x_3} = f \text{ in } Q \tag{7}$$

$$\theta(x, 0) = \theta_0(x) \text{ in } \Omega, \ \theta(x, t) = \theta_g(x, t) = C^*(g(x, t)) \text{ on } \Sigma.$$

We can consider now two approaches. If we would like to catch the occurrence and development of saturation ($h \geq 0$ or $\theta = \theta_s$) we take into account model (7) with $\beta^*(\theta)$ the multivalued operator given by (5) with the conditions $(i) - (ii), (ii_K)$.

If the interest is in the evolution in the unsaturated situation only ($h < 0$, $\theta < \theta_s$) we shall study the model (7) in the domain $\theta < \theta_s$, where β^* satisfies $(i), (ii)$ but blows up at $\theta = \theta_s$, i.e.,

(iii) $\lim\limits_{\theta \to \theta_s} \beta^*(\theta) = \infty.$

2. Unsaturated model ($\theta < \theta_s$).

Consider again model (7) with conditions $(i) - (iii), (ii_K)$. Moreover we assume that $\theta_g(x, t) < \theta_s$, $\forall (x, t) \in \Gamma \times [0, T]$, expressing the fact that we admit that the saturation does not reach the boundary. Previous results on this problem (for the case $\beta(\theta) < \infty$) have been given in [7], [1], [4], [8], [9].

Functional framework. For convenience we shall denote by $(,)$ and $\|.\|$ the scalar product and respectively the norm in $L^2(\Omega)$. Also, if any confusion is avoided we shall no longer indicate in the integrands those function arguments that represent the integration variables.

Let us consider the spaces $V = H_0^1(\Omega)$, with the usual scalar product and its dual $V' = H^{-1}(\Omega)$. On V' we define the scalar product by $< u, \overline{u} >_{V'} = (u, \psi)$, with $\psi \in V$ the solution of the problem $-\Delta \psi = \overline{u}$, $\psi = 0$ on Γ.

Suppose that there exists a function w with the properties:

($\mathbf{H_g}$) $w \in L^2(0,T;H^1(\Omega)) \cap L^\infty(Q)$, $w_t \in L^2(\Omega \times (0,T))$, $\|w\|_{L^\infty(Q)} < \theta_s$, and $w = \theta_g$ on $\Gamma \times [0,T]$.

Now, by replacing θ by $\phi = \theta - w$, we shall reduce (7) to the problem with a homogeneous boundary condition

$$\frac{\partial \phi}{\partial t} - \Delta\beta^*(\phi + w) + \frac{\partial \kappa(\phi + w)}{\partial x_3} = f - \frac{\partial w}{\partial t} \text{ in } Q \tag{8}$$

$$\phi(x,0) = \theta_0(x) - w(x,0) \overset{notation}{=} \phi_0(x) \text{ in } \Omega,\ \phi(x,t) = 0 \text{ on } \Sigma.$$

We define the operator $A(t) : V \to V'$ by

$$(A(t)\phi, \psi) = \int_\Omega \nabla\beta^*(\phi + w)\cdot\nabla\psi dx - \int_\Omega \frac{\partial \kappa(\phi + w)}{\partial x_3}\psi dx,\ \forall \psi \in V$$

and so we are led to the Cauchy problem

$$\frac{d\phi}{dt} + A(t)\phi = f - \frac{\partial w}{\partial t} \text{ a.e. } t \in (0,T), \tag{9}$$

$$\phi(0) = \phi_0(x) \text{ in } \Omega. \tag{10}$$

Easily one can show that if ϕ is a strong solution to the Cauchy problem (9)-(10) then it satisfies (8) in the sense of distributions. It is convenient to rewrite (9) in the following form

$$\frac{d\phi}{dt} + B(t)\phi = f^B - \frac{dw}{dt} \text{ a.e. } t \in (0,T) \tag{11}$$

where the time dependent operator $B(t) : V \to V'$ is defined by

$$(B(t)\phi, \psi) = \int_\Omega \nabla F^w(\phi + w)\cdot\nabla\psi dx - \int_\Omega \kappa(\phi + w)\frac{\partial \psi}{\partial x_3} dx,\ \forall \psi \in V, \tag{12}$$

$$F^w(\phi) = \beta^*(\phi + w) - \beta^*(w),\ \forall \phi \in V,\ f^B = f + \Delta\beta^*(w). \tag{13}$$

2.1. The approximating problem. We have to face with the fact that the function β^* blows up at the saturation value θ_s, reason for which we first shall consider an approximating problem. We take $\varepsilon > 0$ and we approximate the function $\beta^*(\theta)$ in the following way

$$\beta^*_\varepsilon(\theta) = \begin{cases} \beta^*(\theta) \text{ if } \theta \le \theta_s - \varepsilon \\ \beta^*(\theta_s - \varepsilon) + (\theta - \theta_s + \varepsilon)\beta(\theta_s - \varepsilon) \text{ if } \theta > \theta_s - \varepsilon. \end{cases} \tag{14}$$

We remind that for $\theta < \theta_s$ the function β is monotonically increasing and for each $\varepsilon > 0$ we have $\rho \le \beta_\varepsilon(\theta) \le K_\varepsilon = K_s/C((C^*)^{-1}(\theta_s - \varepsilon))$. So we write the approximating problem in the form

$$\frac{d\phi_\varepsilon}{dt} + B_\varepsilon(t)\phi = f^B - \frac{dw}{dt} \text{ a.e. } t \in (0,T) \tag{15}$$

$$\phi_\varepsilon(0) = \phi_0 \tag{16}$$

where $\phi_\varepsilon = \theta_\varepsilon - w$, $F_\varepsilon(\phi) = \beta^*_\varepsilon(\phi + w) - \beta^*_\varepsilon(w) \in H^1_0(\Omega), \forall \phi \in V$ and the operator $B_\varepsilon(t) : V \to V'$ is defined by

$$(B_\varepsilon(t)\phi, \psi) = \int_\Omega \nabla F_\varepsilon(\phi) \cdot \nabla \psi dx - \int_\Omega \kappa(\phi + w) \frac{\partial \psi}{\partial x_3} dx, \ \forall \psi \in V.$$

Properties of the operator $B_\varepsilon(t)$. For each $\varepsilon > 0$, The operator $B_\varepsilon(t)$ is continuous from V to V' and satisfies

$$\|B_\varepsilon(t)\phi\|_{V'} \le \alpha_\varepsilon \|\phi\|_V + \alpha^1_\varepsilon, \ \forall \phi \in V,$$
$$(B_\varepsilon(t)\phi, \phi) \ge \frac{\rho}{4} \|\phi\|^2_V - \gamma_0 \|\phi\|^2 - \gamma_\varepsilon,$$

where $\alpha_\varepsilon = \sigma_\varepsilon + Mc_\Omega$, $\alpha^1_\varepsilon = \sigma^1_\varepsilon + M\|w\| + c_M$, $\sigma_\varepsilon = c_\Omega K_\varepsilon$, $\sigma^1_\varepsilon = (K_\varepsilon + K_w)\|w\|$, c_M follows from (i_K) and c_Ω is the constant occurring in Friedrich's inequality. Also $\gamma_0 = M^2/\rho$, $\gamma_\varepsilon = (q^2_\varepsilon + M^2)/\rho \|w\|_{H_1(\Omega)} + c^2_M/\rho$, $q_\varepsilon = K_\varepsilon + K_w$, $K_w = K((C^*)^{-1}(w))$.

Let $j : \mathbb{R} \to (-\infty, \infty]$ by defined by

$$j(r) = \begin{cases} \int_0^r \beta^*(\xi)d\xi, \ r \le \theta_s \\ +\infty, \ r > \theta_s. \end{cases} \tag{17}$$

Then, (see [6]), j is a proper, convex and lower-semicontinuous (l.s.c.) function on $\mathbb{R}$ and $\partial j(r) = \beta^*(r)$.

PROPOSITION 1. *Assume hypothesis* $(\mathbf{H_g})$, *let* $f \in L^2(0,T; H^{-1}(\Omega))$ *and* $\theta_0 \in L^2(\Omega)$. *Then the approximating problem* (15)-(16) *has, for each* $\varepsilon > 0$, *a unique solution* ϕ_ε *absolutely continuous on* $[0,T]$, *that satisfies*

$$\phi_\varepsilon \in L^2(0,T; H^1_0(\Omega)), \ d\phi_\varepsilon/dt \in L^2(0,T; H^{-1}(\Omega)), \tag{18}$$

implying that

$$\theta_\varepsilon \in L^2(0,T; H^1(\Omega)), \ d\theta_\varepsilon/dt \in L^2(0,T; H^{-1}(\Omega)). \tag{19}$$

Moreover, ϕ_ε *satisfies the estimate*

$$\frac{\rho}{2} \|\theta_\varepsilon(t)\|^2 \le \int_\Omega j_\varepsilon(\theta_\varepsilon(t))dx + \int_0^t \left\| \frac{d(\phi_\varepsilon + w)(\tau)}{d\tau} \right\|^2_{V'} d\tau + \tag{20}$$

$$+ \int_0^t \|F_\varepsilon(\phi_\varepsilon(\tau))\|^2_V d\tau \le c_0 (\int_\Omega j(\theta_0)dx + \int_0^t \|f^B(\tau)\|^2_{V'} d\tau + 1), \ \forall t \in (0,T).$$

Here c_0 *is a constant independent on* ε *and* $j_\varepsilon(r) = \int_0^r \beta^*_\varepsilon(\xi)d\xi$.

THEOREM 1. *Let us assume hypothesis* $(\mathbf{H_g})$ *and*

$$f \in L^2(0,T; H^{-1}(\Omega)), \ \theta_0 \in L^2(\Omega), \ j(\theta_0) \in L^1(\Omega). \tag{21}$$

Then there exists a unique strong solution ϕ *to* (9)-(10) *that satisfies*

$$\phi \in L^2(0,T; H^1_0(\Omega)), \ d\phi/dt \in L^2(0,T; H^{-1}(\Omega)), \tag{22}$$

$$\theta \in L^2(0,T; H^1(\Omega)), \ d\theta/dt \in L^2(0,T; H^{-1}(\Omega)). \tag{23}$$

Moreover, it follows that

$$\beta^*(\theta) \in L^2(0,T;H^1(\Omega)),\ j(\theta) \in L^\infty(0,T;L^1(\Omega)). \tag{24}$$

If the initial solution is less regular, i.e. $\theta_0 \in M_{\theta_s} = \{\theta \in L^2(\Omega); \theta \leq \theta_s\}$, which represents the closure in V' of the set $M_j = \{\theta \in L^2(\Omega);\ j(\theta) \in L^1(\Omega)\}$ we can enounce the next result

THEOREM 2. *Let* $f \in L^2(0,T;H^{-1}(\Omega))$, $\theta_0 \in M_{\theta_s}$ *and assume* $(\mathbf{H_g})$. *Then the Cauchy problem* (9)-(10) *has a unique solution* $\theta \in C([0,T], H^{-1}(\Omega))$ *such that*

$$\theta \in W^{1,2}(\delta,T;H^{-1}(\Omega)) \text{ for every } 0 < \delta < T, \tag{25}$$

$$j(\theta) \in L^1(Q), \tag{26}$$

$$\sqrt{t}d\theta/dt \in L^2(0,T;H^{-1}(\Omega)),\ \sqrt{t}D^*(\theta) \in L^2(0,T;H^1(\Omega)). \tag{27}$$

2.2. Proof of the results. Since $B_\varepsilon(t)$ satisfies the properties indicated before and $f^B \in L^2(0,T;V')$, (18)-(19) follow from a well known theorem of Lions, [5]. To obtain estimate (20) we multiply equation (15) by $F_\varepsilon(\phi_\varepsilon) \in H_0^1(\Omega)$ and integrate over $\Omega \times (0,t)$ for $t \in (0,T)$

$$\int_0^t \int_\Omega \frac{d(\phi_\varepsilon + w)}{d\tau} F_\varepsilon(\phi_\varepsilon) dx d\tau + \int_0^t \int_\Omega |\nabla F_\varepsilon(\phi_\varepsilon)|^2 dx d\tau =$$

$$= \int_0^t \int_\Omega f^B F_\varepsilon(\phi_\varepsilon) dx d\tau + \int_0^t \int_\Omega \kappa(\phi_\varepsilon + w) \frac{\partial F_\varepsilon(\phi_\varepsilon)}{\partial x_3} dx d\tau.$$

We take into account the relationships

$$\frac{\partial j_\varepsilon(\phi_\varepsilon)}{\partial t} = \beta_\varepsilon^*(\phi_\varepsilon)\frac{\partial \phi_\varepsilon}{\partial t},\ j_\varepsilon(r) = \int_0^r \int_0^\xi \beta_\varepsilon(\sigma) d\sigma d\xi \geq \frac{\rho}{2} r^2$$

$$j_\varepsilon(\theta_0) = \int_0^{\theta_0} \beta_\varepsilon^*(\xi) d\xi \leq j(\theta_0).$$

After some calculations we find that

$$\frac{\rho}{2}\|\theta_\varepsilon(t)\|^2 \leq \int_\Omega j_\varepsilon(\theta_\varepsilon(t))dx + \frac{1}{4}\int_0^t \|F_\varepsilon(\phi_\varepsilon(\tau))\|_V^2 d\tau \leq$$

$$\leq c_S + (M^2 + \frac{1}{2})\int_0^t \|\theta_\varepsilon(\tau)\|^2 d\tau + \frac{\rho}{4}\|\theta_\varepsilon(t)\|^2,$$

where

$$c_S = \int_\Omega j(\theta_0)dx + \frac{1}{2}\|\theta_0\|^2 + \frac{1}{2}\|\beta_\varepsilon^*(w_0)\|^2 + \frac{1}{2}\int_0^t \left\|\beta_\varepsilon(w(\tau))\frac{\partial w}{\partial \tau}\right\|^2 d\tau +$$

$$+ \int_0^t \left\|f^B(\tau)\right\|_{V'}^2 d\tau + c_M^2 t + \frac{1}{\rho}\|\beta_\varepsilon^*(w(t))\|^2 < \infty.$$

Using Gronwall's lemma we obtain that

$$\int_\Omega j_\varepsilon(\theta_\varepsilon(t))dx + \frac{1}{4}\int_0^t \|F_\varepsilon(\phi_\varepsilon(\tau))\|_V^2 d\tau \leq c_1(\int_\Omega j(\theta_0)dx + \int_0^t \left\|f^B(\tau)\right\|_{V'}^2 d\tau + 1).$$

Then we multiply (15) scalarwise in V' by $d(\phi_\varepsilon + w)/dt$ and by similar calculations we get that

$$\int_\Omega j_\varepsilon(\theta_\varepsilon(t))dx + \frac{1}{4}\int_0^t \left\| \frac{d\theta_\varepsilon(\tau)}{d\tau} \right\|_V^2 d\tau \le c_2(\int_\Omega j(\theta_0)dx + \int_0^t \left\| f^B(\tau) \right\|_{V'}^2 d\tau + 1).$$

By adding these last results we obtain (20) as claimed. The constants c_1 and c_2 are independent on ε. The requirement that $\theta_g(x,t) < \theta_s$, $\forall(x,t) \in \Gamma \times [0,T]$ ensures that $\beta_\varepsilon(w) = \beta(w) < \infty$ as necessary in this proof.□

Proof of Theorem 1. We let ε to tend to 0. If $j(\theta_0) \in L^1(\Omega)$, we derive from (20) that $\{\phi_\varepsilon\}$ lies in a bounded subset of $L^\infty(0,T;L^2(\Omega))$, $\{d\phi_\varepsilon/dt\}$ lies in a bounded subset of $L^2(0,T;H^{-1}(\Omega))$ and $\{F_\varepsilon(\phi_\varepsilon)\}$ is in a bounded subset of $L^2(0,T;H_0^1(\Omega))$. Using (i) we get that $(F_\varepsilon)^{-1}$ is Lipschitz and hence (ϕ_ε) is in a bounded subset of $L^2(0,T;H_0^1(\Omega))$, too. Since $V = H_0^1(\Omega)$ is compactly embedded in $H = L^2(\Omega)$ we conclude, according to Lions-Aubin theorem, [5] that $\{\phi_\varepsilon\}$ is compact in $L^2(0,T;L^2(\Omega))$, i.e., on a subsequence

$$\phi_\varepsilon \to \phi \text{ in } L^2(0,T;L^2(\Omega)) \text{ as } \varepsilon \to 0 \, .$$

So, we conclude that there exists a subsequence (that will be denoted ϕ_ε too) such that

$$\phi_\varepsilon \to \phi \text{ weakly in } L^2(0,T;H_0^1(\Omega)),$$

$$d\phi_\varepsilon/dt \to d\phi/dt \text{ weakly in } L^2(0,T;H^{-1}(\Omega)),$$

$$F_\varepsilon(\phi_\varepsilon) \to \chi \text{ weakly in } L^2(0,T;H_0^1(\Omega)),$$

implying that

$$\beta_\varepsilon^*(\phi_\varepsilon + w) \to \eta \text{ weakly in } L^2(0,T;H^1(\Omega)).$$

Then it also follows that $\phi_\varepsilon(x,t) \to \phi(x,t)$ a.e. on $\Omega \times (0,T)$. We shall prove that $\eta = \beta^*(\phi + w)$. If $\phi_\varepsilon(x,t)$ converges a.e. to $\phi(x,t)$, we have, using Egorov's theorem that for each $\delta > 0$, there exists a measurable subset $Q_\delta \subset Q$, with meas$(Q \backslash Q_\delta) < \delta$ and $\phi_\varepsilon \to \phi$ uniformly on Q_δ. Since β_ε^* is continuous and $\beta_\varepsilon^*(r) \to \beta^*(r)$, as $\varepsilon \to 0$ on $\{r; r < \theta_s\}$, we have that $\beta_\varepsilon^*((\phi_\varepsilon + w)(x,t)) \to \beta^*((\phi + w)(x,t))$ on Q_δ and hence $\beta_\varepsilon^*(\phi_\varepsilon + w) \to \beta^*(\phi + w)$ weakly in $L^2(Q_\delta)$. Then we get that $\eta = \beta^*(\phi + w)$ a.e. on Q. Therefore

$$\chi = \lim_{\varepsilon \to 0}(\beta_\varepsilon^*(\phi + w) - \beta_\varepsilon^*(w)) = \beta^*(\phi + w) - \beta^*(w) = F^w(\phi)$$

weakly in $L^2(0,T;H_0^1(\Omega))$. Now due to the continuity of the operators $\partial/\partial x_3$ and $-\Delta$ we have that $B_\varepsilon(t)\phi_\varepsilon \to B(t)\phi$ weakly in $L^2(0,T;H^{-1}(\Omega))$. Now we can pass to limit as $\varepsilon \to 0$ in equations (15)-(16) and obtain that

$$\frac{d\phi}{dt} + B(t)\phi = f^B - \frac{dw}{dt} \text{ a.e. } t \in (0,T), \; \phi(0) = \phi_0.$$

Moreover from (20) it follows that $j_\varepsilon(\theta_\varepsilon) \to j(\theta)$ weak star in $L^\infty(0,T;L^1(\Omega))$ completing so the proof of the Theorem 1.□

Proof of Theorem 2. Consider first $f \in L^2(0,T;H^{-1}(\Omega))$ and $\theta_0 \in M_j$. So we can apply Theorem 1 to find that the Cauchy problem (9)-(10) has a unique solution $\phi \in C([0,T],H^{-1}(\Omega))$ that satisfies (21)-(23). Next the idea is to multiply scalarwise in V'

equation (9) directly by $td\theta/dt$ and integrate over $(0,T)$. Some calculations, using the appropriate techniques from [2] and [3], lead to the estimate

$$\int_0^T t\left\|\frac{d\theta}{dt}(t)\right\|_{V'}^2 dt \le \gamma(\|\theta_0\|_{V'}^2 + \int_0^T t\,\|f(t)\|_{V'}^2\,dt + 1), \tag{28}$$

Similarly, multiplying scalarly in V' equation (9) by θ and integrate over $(0,T)$ we get

$$\|\theta(t)\|_{V'}^2 + \int_0^t \|\theta(\tau)\|^2\,d\tau \le \alpha_0\,\|\theta_0\|_{V'}^2 + \alpha_1\int_0^t \|f(\tau)\|_{V'}^2\,d\tau. \tag{29}$$

Also, like in Theorem 1 we still obtain $\int_0^T t\,\|F^w(\phi(t))\|_V^2 <$ constant. Detailed calculations can be found in [6].

Further we take $\theta_0 \in M_{\theta_s}$ and $f \in L^2(0,T;V')$. Then there exists $\{\theta_n^0\} \subset M_j$ such that $\theta_n^0 \to \theta_0$ in V'. Hence problem (15)-(16) with the initial condition θ_n^0 has, by Theorem 1, a solution θ_n satisfying (21)-(23), (28)-(29) and its limit for $n \to \infty$ satisfies (9)-(10).

1. *Alt H. W., Luckhaus S., Visintin A.* On nonstationary flow through porous media // Ann. Mat. Pura Appl.- vol. 136.- 1984.- p. 303-316.
2. *Barbu V.* Nonlinear semigroups and differential equations in Banach spaces // Editura Academiei-Noordhoff International Publishing.- Bucureşti-Leyden.- 1976.
3. *Brezis H.* Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert.- North Holland.- 1973.
4. *Gilardi G.* A new approach to evolution free boundary problems // Comm. Part. Diff. Equ.- vol. 4.- 1979.- p. 1099-1122.
5. *Lions J.L.* Quelques méthodes de résolution des problèmes aux limites non linéaires // Dunod.- Paris.- 1969.
6. *Marinoschi G.* Nonlinear infiltration with a singular diffusion coefficient // Differential and Integral Equations.- v. 16.- no. 9.- 2003.- p. 1093-1110.
7. *Oleinik O.A., Kalashnikov A.S., Chzou Yui-Lin* The Cauchy problem and boundary value problems for equations of the type of nonstationary filtration // Izv. Akad. Nauk. SSSR Ser. Mat.- vol. 22.- 1958.- p. 667-704.
8. *Torelli A.* Su un problema a frontiera libera di evoluzione // Boll. Un. Mat. Ital.- vol. 11.- 1975.- 559-570.
9. *Visintin A.* Existence results for some free boundary filtration problems // Ann. Mat. Pura Appl.- vol. 124.- 1980.- p. 293-320.

Institute of Mathematical Statistics and Applied Mathematics,
Bucharest, Romania
`gmarino@acad.ro`

Received 3.11.2003